NRL Report 6406

Mechanical Impedance and Mobility Concepts

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July 29, 1966

NAVAL RESEARCH LABORATORY
Washington, D.C.

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Abstract: The purpose of this report is to discuss fundamental concepts involved in mechanical impedance and mobility. The equations used in the formulation of the concepts are examined as to meaning and content. It is demonstrated for any prescribed structure that every possible observed mobility element is independent of the number and location of the other measurement points and that every possible observed mechanical impedance element is not. The effects upon the mobility and impedance arrays are examined for simple expansions and contractions of the tensor. Since mobility is not affected by changes in the number of observation points, remeasurement is not required at the observation points that are retained during expansion or contraction of the tensor; but since any changes in the observation points for impedance measurements requires changes in the blocking forces, the observation points that are retained are affected. An undamped lumped parameter system (a three-mass system on a rigid base and constrained to unidirectional motion) serves to illustrate the effects on the tensor elements of impedance or of mobility of the choice of the number and location of measurement points.

INTRODUCTION

Mechanical impedance and mobility concepts became popular when mechanical vibration problems were attacked by drawing an analogous electric circuit to take advantage of the powerful, well-developed techniques of electric circuit theory. There immediately came to the fore a problem of choice between two analogies. Force analogous to current results in a mechanical mobility analog, whereas force analogous to voltage results in a mechanical impedance analog.

The mobility form of the analogy was strongly advocated by Firestone (1) in 1933. He pointed out that the impedance form of the analogy lacked completeness in the laws for combining series and parallel elements, as well as in Kirchoff's laws. Notwithstanding this, mechanical impedance apparently has become the more popular of the two concepts.

On June 2, 1965, a report (2) was presented to a Washington meeting of the Shock and Vibration Committee of the Acoustical Society of America. The discussion revealed strong differences of opinion about fundamental aspects of the analogies and a deep-seated interest in them by both research and engineering personnel. To clarify points raised during the discussion, those ideas are formulated here in greater detail and illustrated by numerical examples.

It is the purpose of this report to discuss the fundamental aspects of the two concepts. The basic equations are examined and explained. Distributed parameter systems are discussed. Specific examples using a lumped parameter model are used to illustrate the strong interdependency of impedance elements upon the number and location of all points of interest. It is shown that this same interdependency does not exist in the case of mobility elements. Therefore, mobilities are invariant properties of a particular structure, while impedances are not. In other words the impedance element z_{ij} depends on the number of other observation points and the mobility element m_{ij} does not.

This report deals only with solid mechanics. The problems of hydraulics and gases are left to their respective specialists.

NRL Problem F02-18; Projects SF 013-10-01-1790 and RR 009-03-45-5757. This is an interim report on one phase of the problem; work is continuing. Manuscript submitted March 2, 1966.

FUNDAMENTALS OF MOBILITY AND IMPEDANCE

Assumptions

For the purposes of this report it is assumed that: linear elastic structures are being discussed, so that superposition holds; the normal rules of usage for a tensor hold; all forces and motions are sinusoidal at the same frequency; and the tensor elements account for the phase as well as the amplitude.

As used herein a measurement is defined as the complex ratio of any two sinusoidal signals. This measurement can be thought of as being displayed in the form of a complex number or as an absolute amplitude and associated phase angle. For example, the measurement of a mobility or impedance element will be thought of as a result which involves the ratio of two vectors.

The terms force and velocity are used in the generalized sense: A force can be a force or a moment, and velocity can be translational or rotational velocity.

Definitions

The principle of superposition will be used to define the quantities called impedance, mobility, and pseudoimpedance.

The set of mobility elements m_{ij} is defined in the following sense:

$$v_i = \sum_j m_{ij} f_j.$$

The set of impedance elements z_{ij} is defined in the following sense:

$$f_i = \sum_j z_{ij} v_j.$$

The set of pseudoimpedance elements Z_{ij}^* is defined in the following sense:

$$v_i = \sum_j \frac{f_j}{Z_{ij}^*}.$$

The three subsections of this report immediately following expand these definitions and discuss the concepts more completely. A discussion subsection then follows which compares the consequences of these three definition equations.

Mobility

Mobility is a tensor (or tensor component) which operationally describes the effects upon the resultant velocity (or several velocities) of the application of a force or an array of forces. The concept of mobility can be represented by the matrix equation

$$V = MF, \tag{1}$$

where V is a column vector of *resultant velocities* v_i, F is a column vector of *applied forces* f_j, and M is a symmetric tensor of mobilities m_{ij}. In expanded form it looks like

$$\begin{aligned} v_1 &= m_{11} f_1 + m_{12} f_2 + m_{13} f_3 + \dots, \\ v_2 &= m_{21} f_1 + m_{22} f_2 + m_{23} f_3 + \dots, \\ v_3 &= m_{31} f_1 + m_{32} f_2 + m_{33} f_3 + \dots, \\ v_4 &= m_{41} f_1 + m_{42} f_2 + \dots, \\ &\mathit{etc.} \end{aligned} \tag{2}$$

Note that $m_{ij} f_j$ defines a velocity at i caused by a force at j. Let this velocity be called $\bar{v}_{ij}$. Then

$$v_i = \sum_j \bar{v}_{ij}. \tag{3}$$

Mobility is thus a concept which sums velocity response.

To measure the elements of the M array: (a) the *forces* are *applied one at a time* to each point of interest, (b) the structure is allowed to respond as it chooses, and (c) the individual elements are measured as the complex ratio of the particular velocity response to the single exciting force. For example, if only f_2 were applied, Eqs. (2) would reduce to the set

$$\begin{aligned} \bar{v}_{12} &= m_{12} f_2, \\ \bar{v}_{22} &= m_{22} f_2, \\ \bar{v}_{32} &= m_{32} f_2, \\ &\mathit{etc.}, \end{aligned} \tag{4}$$

since

$$f_k = 0,\ k \neq 2,$$

and then the complex ratio of $\bar{v}_{i2}$ to f_2 defines m_{i2}:

$$\begin{aligned} m_{12} &= \frac{\bar{v}_{12}}{f_2}, \\ m_{22} &= \frac{\bar{v}_{22}}{f_2}, \\ &\mathit{etc.} \end{aligned} \tag{5}$$

By the ordinary reciprocity theorems of vibrations, $m_{ij} = m_{ji}$.

Impedance

The discussion of the concept of impedance is phrased and organized just like the previous one on mobility:

Impedance is a tensor (or tensor component) which operationally describes the effects upon the resultant force (or several forces) of the application of a velocity or an array of velocities. The concept of impedance can be represented by the matrix equation

$$F = ZV, \tag{6}$$

where F is a column vector of *resultant forces* f_i, V is a column vector of *applied velocities* v_j, and Z is a symmetric tensor of impedances z_{ij}. In expanded form it looks like

$$\begin{aligned} f_1 &= z_{11} v_1 + z_{12} v_2 + z_{13} v_3 + \ldots, \\ f_2 &= z_{21} v_1 + z_{22} v_2 + z_{23} v_3 + \ldots, \\ f_3 &= z_{31} v_1 + z_{32} v_2 + z_{33} v_3 + \ldots, \\ f_4 &= z_{41} v_1 + z_{42} v_2 + \ldots, \\ &\mathit{etc.} \end{aligned} \tag{7}$$

Note that $z_{ij}\, v_j$ defines a force at i caused by a velocity at j. Let this force be called $\bar{f}_{ij}$. Then

$$f_i = \sum_j \bar{f}_{ij}. \tag{8}$$

Impedance is thus a concept which sums force response.

To measure the elements of the Z array: (a) the *velocities* are *applied one at a time* to each point of interest, (b) the structure is not allowed to respond as it chooses, since certain motions corresponding to points where other velocities will be applied are constrained to vanish, and (c) the individual elements are measured as the complex ratio of the particular force response to the single exciting velocity. For example if only v_2 were applied (the remaining points having been constrained), Eqs. (7) would reduce to the set,

$$\begin{aligned} \bar{f}_{12} &= z_{12}\, v_2, \\ \bar{f}_{22} &= z_{22}\, v_2, \\ \bar{f}_{32} &= z_{32}\, v_2, \\ &\mathit{etc.} \end{aligned} \tag{9}$$

since

$$v_k = 0,\ k \neq 2.$$

Now $\bar{f}_{j2}$ $(j \neq 2)$ is the blocking force at j, during excitation by a velocity at 2, which is necessary to constrain the velocity at j to zero, and $\bar{f}_{22}$ is the force which results from the motion at point 2. The complex ratio of $\bar{f}_{j2}$ to v_2 defines z_{j2}:

$$z_{j2} = \frac{\bar{f}_{j2}}{v_2}. \tag{10}$$

It is also well known that $z_{ij} = z_{ji}$.

It is obvious that since $Z = M^{-1}$ (and hence $z_{ik} \neq m_{ik}^{-1}$ except in the trivial case of only one point), the impedance elements can be calculated from measurements which were obtained without using blocking forces. However, this is equivalent to measuring mobilities and calculating impedances, not measuring impedances.

Pseudoimpedance

Some authors (3) define and use something called impedance which is different from the definition in the preceding subsection. This will be called pseudoimpedance in this report and symbolized as Z^*. It is commonly defined as follows:

Let the force F_j, and the velocity V_k be expressed as

$$F_j = \bar{F}_j\, e^{j\omega t+\theta} \tag{11}$$

and

$$V_k = \bar{V}_k\, e^{j\omega t}, \tag{12}$$

where $\bar{F}_j$ and $\bar{V}_k$ are the magnitudes of the respective signals, ω is the exciting frequency, t is time, and θ is the phase. Then

$$Z^*_{kj} = \frac{\bar{F}_j}{\bar{V}_k}\, e^{j\theta}. \tag{13}$$

Two types of mechanical pseudoimpedance can be considered: driving-point pseudoimpedance and transfer pseudoimpedance. If the velocity is obtained at the point of application of the exciting force, the resulting ratio is called driving-point pseudoimpedance, Z^*_{kk}. If the velocity is obtained at some other point on the structure the ratio is referred to as transfer pseudoimpedance, Z^*_{kj}.

If a system has two or more exciting forces, the total velocity at any point is the vectorial sum of the velocities resulting from each force separately. Then

$$V_j = \sum_k \bar{v}_{jk} = \sum_k \frac{\bar{F}_k}{Z^*_{jk}}. \tag{14}$$

Pseudoimpedance may also be expressed by complex numbers.

Examination of Eqs. (2), (3), and (14) shows

$$\begin{aligned} v_1 &= m_{11} f_1 + m_{12} f_2 + m_{13} f_3 + \dots = \frac{\bar{F}_1}{Z^*_{11}} + \frac{\bar{F}_2}{Z^*_{12}} + \frac{\bar{F}_3}{Z^*_{13}} + \dots, \\ v_2 &= m_{21} f_1 + m_{22} f_2 + m_{23} f_3 + \dots = \frac{\bar{F}_1}{Z^*_{21}} + \frac{\bar{F}_2}{Z^*_{22}} + \frac{\bar{F}_3}{Z^*_{23}} + \dots, \end{aligned} \tag{15}$$

etc.

Now if $f_i = \bar{F}_i$ and $m_{ij} = (Z^*_{ij})^{-1}$, the pseudoimpedance is the scalar multiplicative inverse of the mobility, element by element, and not the tensor inverse. Then

$$z_{ik} \neq Z^*_{ik} = (m_{ik})^{-1}. \tag{16}$$

Undoubtedly the fact that there are two commonly accepted different kinds of impedance has caused difficulties. For the rest of this report, pseudoimpedance will be ignored.

Discussion

In the deliberation which follows an attempt will be made to expand upon the definitions of impedance and mobility and to discuss the concepts more fully.

First consider mobility measurement. A single force is applied, and an array of ratios of velocities responding to this single force is measured. The structure has not been artificially constrained. No special effort need be made to apply other external forces to the points of interest during the measurement run. Ignoring feedback from the measuring transducers into the system, observations made anywhere on the system do not affect one another, and the mobility element m_{ij} remains the same whether or not observations are made at n other points. Therefore each mobility element is invariant with respect to the schedule of observations, since there are no artificial constraints. It is only dependent upon its own location and the location of the driving force.

Second, consider analogous straightforward impedance measurements using blocking forces. A single velocity is applied, and an array of ratios of forces responding to this single exciting velocity is measured. The structure has been deliberately constrained by blocking forces which maintain the velocity at zero at all the points scheduled for observation of their respective impedance elements. It is obvious that the constraints imposed by the blocking forces at the points chosen for measurement will affect the response of the structure, and indeed the response will change if either the number or location of the blocking forces is changed. The impedance elements observed will therefore depend upon the particular set of blocking forces used during their observation and hence cannot be termed invariant with respect to the schedule of observations.

The experimental consequences of the fact that mobility elements are invariant with respect to the schedule of measurements and impedance measurements lack this property may be clarified by an example. Suppose there exists a structure which is to be measured by vibration means. It is first decided that observations at four points will meet the needs.

Experimenter A proceeds to measure ten different mobility elements (the matrix is symmetric), and he obtains

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}. \tag{17}$$

Experimenter B prefers the impedance analog and use of blocking forces, so he measures the ten different impedance elements in

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix}. \tag{18}$$

The procedure used by B will be more demanding than that used by A but is still straightforward and feasible if the physical problems of measuring blocked forces are ignored.

Suppose that after the measurements were completed it was decided that five points (including the previous four) should have been used, and the extra information is requested.

Experimenter A fills in the fifth row and column of the mobility array by driving at the new point and making five new measurements. He retains his original array because it is still valid. Mobility elements are invariant to such an expansion of the tensor.

Experimenter B will have to use a new blocking force which was not present before. This new blocking force will also alter the other responses, so he will have to measure all fifteen different impedance elements. This again is straightforward but really seems to be doing things the hard way—all this because impedance elements are not invariant in such an expansion of the tensor.

Suppose reconsideration now shows that only three of the five points are going to be used. Experimenter A supplies the desired mobility elements immediately by ignoring the extraneous rows and columns, since the mobility elements are invariant in this kind of contraction. Experimenter B would have to measure the six different impedance elements to form the new three-by-three array, since the impedance elements are not invariant in this kind of contraction.

It seems reasonable to conclude this comparison with the remark that both concepts are useful but the invariance of the mobility elements seems to be attractive. In this sense perhaps mobility could be called the natural tensor and impedance the derived tensor.

A LUMPED PARAMETER SYSTEM

An example of an undamped lumped parameter system is used here to illustrate the effects of the previous discussion. A three-mass system on a rigid base and constrained to unidirectional motion has been chosen as the example because of the ease with which the reader might follow along. The schematic of the model and the necessary mathematical manipulations are found in the Appendix.

The reader is asked to realize that the purpose of this example is only to illustrate the effects upon the impedance elements of the initial choice of the number of points where measurements are to be taken. For example, it is assumed here that no points on the massless springs are going to be used, so that, from the possibility of an infinite set of points to consider, the problem has been reduced to a maximum number of three points of possible consideration.

Suppose four investigators were told only to "measure the direct impedance z_{11}" on this system. Investigator I chooses to measure at all three masses. Investigator II restricts his attention to only masses 1 and 2. Investigator III measures only at points 1 and 3. Investigator IV places his impedance head and all his effort on point 1.

They then compare their results for the impedance element z_{11}, with the results given in Table 1. Note that Investigators I and II agree and might be tempted to say that III and IV must have done something wrong. To check further I and II might compare their results for z_{22} and find to their consternation the results given in Table 2. Yet in each case all the investigators are correct, for if some one were to suggest that they all calculate the mobility coefficient m_{11} from their respective impedance arrays, they would all come to the same conclusion:

$$m_{11} = \frac{j_{\omega}(234 - 246_{\omega}^{2} + 48_{\omega}^{4})}{648 - 2124_{\omega}^{2} + 1272_{\omega}^{4} - 192_{\omega}^{6}}.$$

TABLE 1
Results of Direct Impedance Measurement z_{11}

Investigator	Results for z_{11}
I	$\frac{6 - 4_{\omega}^{2}}{j_{\omega}}$
II	$\frac{6 - 4_{\omega}^{2}}{j_{\omega}}$
III	$\frac{1}{j_{\omega}}\left(\frac{54 - 96_{\omega}^{2} + 24_{\omega}^{4}}{15 - 6_{\omega}^{2}}\right)$
IV	$\frac{1}{j_{\omega}}\left(\frac{648 - 2124_{\omega}^{2} + 1272_{\omega}^{4} - 192_{\omega}^{6}}{234 - 246_{\omega}^{2} + 48_{\omega}^{4}}\right)$

TABLE 2
Results of Direct Impedance
Measurement z_{22}

Investigator	Results for z_{22}
I	$\frac{15 - 6_{\omega}^{2}}{j_{\omega}}$
II	$\frac{1}{j_{\omega}}\left(\frac{234 - 246_{\omega}^{2} + 48_{\omega}^{4}}{21 - 8_{\omega}^{2}}\right)$

This, of course, has been an example of what can happen when the phrase "measure the direct impedance" is used without qualification as to the other factors. The reader might ask at this point: "Why not require that each mass point be accounted for?" This deserves an answer in the form of two questions:

1. What would you do with a real structure which has, of course, distributed mass and elasticity?

2. Why bother, particularly in view of the physical difficulties of applying "blocking forces and moments," when an unambiguous theory, mobility is readily available?

DISTRIBUTED PARAMETER STRUCTURES

An interesting observation can be made concerning impedance, mobility, and distributed parameter structures. Consider a cantilever column with a tensile-compressive force applied at the free end which is well below the buckling limit. Model the column as a lumped parameter system of n equal elements numbered from 1 on the free-end mass to n near the base. Demand that the direct impedance z_{11} be found, accounting for at least the two elements mass 1 and mass 2. Since, when z_{11} is found, masses 2 through n are not moving due to the blocking force at 2, the impedance increases without bound at the low frequency end as the number of masses is increased.

Consider now beams and frames. In addition to "blocking forces," "blocking moments" may have to be used. What happens to the value of the impedance elements as the number of points of observation get very large and they are quite close together?

In the case of mobility and the cantilever column no such difficulty exists because there are no blocking forces, and it does not matter how many stations are desired. The direct mobility is unaffected. For beams and frames similar statements may be made.

SUMMARY

The ordinary tensor definitions of impedance and mobility have been examined in some detail to point out the features of each. It has been demonstrated that the mobilities of a given structure do not interdepend upon both the location and number of points of interest and that impedances do so depend.

An example (using a lumped parameter system for convenience) was used to illustrate the effects upon the tensor elements of impedance, of the choice of the number and location of measurement points.

It is not implied that impedance is a useless concept in the mathematical sense but only that the investigators must be knowledgeable in its use.

It was not the purpose of this report to discuss the physical difficulties inherent in measuring blocking forces and moments. The workers in the field have found this out for themselves.

Mobilities describe invariant characteristics of the whole structure; impedances generally concern themselves only with segments. Impedances are dependent upon the number of observation points considered and consequently do not possess invariant characteristics.

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Appendix

IMPEDANCES AND MOBILITIES OF A THREE DEGREE OF FREEDOM SYSTEM

Consider the undamped lumped-parameter three degree of freedom system illustrated in Fig. A1. It is assumed that the masses are constrained to unidirectional translation and that only three possible points of interest exist, so that the dimensions of the structure may be neglected. These points of interest are assumed to coincide with the centers of gravity of each of the masses, and no points on the massless springs are considered.

It is convenient to write the equations of motion using the mass and stiffness approach, because the values of the spring constants have been supplied. The equations are

$$\begin{aligned} &4\ddot{x}_1 + 6(x_1 - x_2) = f_1, \\ &6\ddot{x}_2 + 6(x_2 - x_1) + 9(x_2 - x_3) = f_2, \\ &8\ddot{x}_3 + 9(x_3 - x_2) + 12x_3 = f_3. \end{aligned} \tag{A1}$$

Assuming all the forces and motions are sinusoidal at frequency ω in radians per unit time yields

$$\begin{aligned} &(6 - 4\omega^2)x_1 - 6x_2 + 0x_3 = f_1, \\ &-6x_1 + (15 - 6\omega^2)x_2 - 9x_3 = f_2, \\ &0x_1 - 9x_2 + (21 - 8\omega^2)x_3 = f_3. \end{aligned} \tag{A2}$$

Writing this for velocities in matrix notation yields

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \frac{1}{j\omega} \begin{bmatrix} (6 - 4\omega^2) & -6 & 0 \\ -6 & (15 - 6\omega^2) & -9 \\ 0 & -9 & (21 - 8\omega^2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{A3}$$

where $\dfrac{(6 - 4\omega^2)}{j\omega} = z_{11}$, $\dfrac{-6}{j\omega} = z_{12}$, *etc.*

To find the mobility form the matrix is inverted, which yields

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{j\omega}{D} \begin{bmatrix} (234 - 246\omega^2 + 48\omega^4) & (126 - 48\omega^2) & 54 \\ (126 - 48\omega^2) & (126 - 132\omega^2 + 32\omega^4) & (54 - 36\omega^2) \\ 54 & (54 - 36\omega^2) & (54 - 96\omega^2 + 24\omega^4) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \tag{A4}$$

where D is the system frequency characteristic given by

$$D = 648 - 2124\omega^2 + 1272\omega^4 - 192\omega^6. \tag{A5}$$

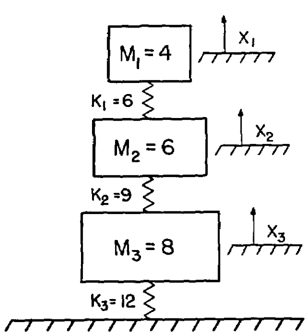

Fig. A1 – Undamped lumped-parameter three degree of freedom system

From (A4) and (A5)

$$m_{11} = \frac{j\omega(234 - 246\omega^2 + 48\omega^4)}{648 - 2124\omega^2 + 1272\omega^4 - 192\omega^6}, \ etc.$$

Before proceeding a formula will be presented for the inversion of a two-by-two symmetric matrix as this will be needed. If

$$A = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{12} & a_{22} \end{bmatrix};$$

then

$$A^{-1} = \frac{1}{a_{11}\,a_{22} - a_{12}^2} \begin{bmatrix} a_{22} & a_{12} \\ a_{12} & a_{11} \end{bmatrix}. \tag{A6}$$

Let us consider the work of four investigators, as described in the main body of the text.

Investigator I chose to measure so as to account for all three possible points of interest. He first applied a velocity to mass 1, measured the force-to-velocity ratio there, and measured the ratios of blocking forces at masses 2 and 3 to the applied velocity. He arrived at

$$z_{11} = \frac{\bar{f}_{11}}{v_1} = \frac{6 - 4\omega^2}{j\omega},$$

$$z_{21} = \frac{\bar{f}_{21}}{v_1} = -\frac{6}{j\omega},$$

$$z_{31} = \frac{\bar{f}_{31}}{v_1} = 0,$$

where $\bar{f}_{21}$ and $\bar{f}_{31}$ are the resultant blocking forces at these points. Note that $\bar{f}_{31}$ is zero in this dynamical chain problem because of the blocking force at point 2. He continued in this fashion applying one velocity at a time and blocking the other two points of interest until he arrived at the results presented in Eq. (A3).

Investigator II was under the impression that only points 1 and 2 were of interest. Therefore mass 3 was allowed to move and no blocking force was applied there. Then from the equations of motion $9v_2 = (21 - 8\omega^2)\,v_3$. Introducing this in the other equations yields

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{1}{j\omega} \begin{bmatrix} (6 - 4\omega^2) & -6 \\ -6 & \dfrac{234 - 246\omega^2 + 48\omega^4}{21 - 8\omega^2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{A7}$$

Note that the z_{22} of Investigator II does not agree with the z_{22} of Investigator I, given in Eq. (A3).

Investigator III measured at points 1 and 3. This means that

$$v_2 = \frac{6v_1}{15 - 6\omega^2} + \frac{9v_3}{15 - 6\omega^2}.$$

Then III obtained

$$\begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \frac{1}{j\omega(15 - 6\omega^2)} \begin{bmatrix} (54 - 96\omega^2 + 24\omega^4) & -54 \\ -54 & (234 - 246\omega^2 + 48\omega^4) \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix}. \tag{A8}$$

His element z_{11} does not agree with I and II.

Investigator IV turned all his attention to point 1 because he thought that only this point was of interest, or perhaps he was thinking of the definition of pseudoimpedance as previously given, and as a consequence he applied no blocking forces. Under these conditions

$$v_3 = \frac{9v_2}{21 - 8\omega^2}$$

and

$$6v_1 = \left[15 - 6\omega^2 - \frac{81}{21 - 8\omega^2}\right] v_2 .$$

Substitution in the equation for f_1 gave the results

$$f_1 = \frac{648 - 2124\omega^2 + 1272\omega^4 - 192\omega^6}{j\omega\ (234 - 246\omega^2 + 48\omega^4)}\ v_1. \tag{A9}$$

Now suppose each investigator were to invert his particular impedance tensor array.

Investigator I would find Eq. (A4).

Investigator II would find

$$Z_{II}^{-1} = \frac{j\omega}{D}\begin{bmatrix} (234 - 246\omega^2 + 48\omega^4) & (126 - 48\omega^2) \\ (126 - 48\omega^2) & (126 - 132\omega^2 + 32\omega^4) \end{bmatrix},$$

which is of course the mobility tensor of Eq. (A4) with the third column and row removed.

Investigator III would find

$$Z_{III}^{-1} = \frac{j\omega}{D}\begin{bmatrix} (234 - 246\omega^2 + 48\omega^4) & 54 \\ 54 & (54 - 96\omega^2 + 24\omega^4) \end{bmatrix},$$

which is of course the mobility tensor of Eq. (A4) with the second column and row removed.

Investigator IV would find

$$Z_{IV}^{-1} = \frac{j\omega\ (234 - 246\omega^2 + 48\omega^4)}{648 - 2124\omega^2 + 1272\omega^4 - 192\omega^6} = m_{11},$$

which is of course the mobility tensor of Eq. (A4) with the second and third rows and columns removed.

This process could be continued to examine the effects of other driving points with various combinations of blocking forces, but this seems redundant, as all the investigators would have been led to the obvious conclusion: every possible mobility element is independent of the number of measurement points, and every possible mechanical impedance element is not.

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

MECHANICAL IMPEDANCE AND MOBILITY CONCEPTS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
An interim report on one phase of the problem.

5. AUTHOR(S) *(Last name, first name, initial)*

O'Hara, G. J.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 29, 1966	16	3

8a. CONTRACT OR GRANT NO. NRL Problem F02-18 b. PROJECT NO. SF 103-10-01-1790 c. RR 009-03-45-5757 d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 6406 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

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13. ABSTRACT

The purpose of this report is to discuss fundamental concepts involved in mechanical impedance and mobility. The equations used in the formulation of the concepts are examined as to meaning and content. It is demonstrated for any prescribed structure that every possible observed mobility element is independent of the number and location of the other measurement points and that every possible observed mechanical impedance element is not. The effects upon the mobility and impedance arrays are examined for simple expansions and contractions of the tensor. Since mobility is not affected by changes in the number of observation points, remeasurement is not required at the observation points that are retained during expansion or contraction of the tensor; but since any changes in the observation points for impedance measurements requires changes in the blocking forces, the observation points that are retained are affected. An undamped lumped parameter system (a three-mass system on a rigid base and constrained to unidirectional motion) serves to illustrate the effects on the tensor elements of impedance or of mobility of the choice of the number and location of measurement points.

DD FORM 1 JAN 64 1473

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Mechanics						
Dynamics						
Mechanical waves						
Force (mechanics)						
Mechanical properties						
Mechanical impedance						
Mechanical mobility						
Shock waves						
Oscillation						
Velocity						
Mathematical analysis						
Structures						
Lumped parameter system						
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